

Parametrices and hypoellipticity for pseudodifferential operators on spaces of tempered ultradistributions

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Abstract

We construct parametrices for a class of pseudodifferential operators of infinite order acting on spaces of tempered ultradistributions of Beurling and Roumieu type. As a consequence we obtain a result of hypoellipticity in these spaces.

0 Introduction

The main concern in this paper is the study of hypoellipticity for pseudodifferential operators in the setting of tempered ultradistributions of Beurling and Roumieu type on \mathbb{R}^d . These distributions represent the global counterpart of the ultradistributions studied by Komatsu, see [12, 13, 16]. We recall that the space of test functions for the ultradistributions of [12, 13, 16] is a natural generalisation of the Gevrey classes. In the same way tempered ultradistributions act on a space which generalises the spaces of type \mathcal{S} introduced by Gelfand and Shilov in [9].

Before presenting our results let us recall some previous results on hypoellipticity in the spaces mentioned above. Hypoellipticity in Gevrey classes has been studied by several authors, see [11, 17, 22, 25] and the references therein. Indeed the functional setting allows to consider very general symbols $a(x, \xi)$ admitting exponential growth at infinity with respect to the covariable ξ . This was first noticed in [25] and generalised in [6, 7] with applications to hyperbolic equations in Gevrey classes. In [25] the hypoellipticity has been obtained by means of the construction of a parametrix. More recently, the results of [25] have been extended by Fernández et al. [8] to the space of ultradistributions of Beurling type and by the first author to the global frame of the Gelfand-Shilov spaces of type \mathcal{S} , see [2, 3, 4], allowing exponential growth for the symbols also with respect to the variable x .

It is then natural to study the same problem for pseudodifferential operators acting on tempered ultradistributions. In a recent paper [21], the third author constructed a global calculus for pseudodifferential operators of infinite order of Shubin type in this setting. Here we want to apply this tool to construct parametrices for the class of [21] and to prove a hypoellipticity result.

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Let us first fix some notation and introduce the functional setting where our results are obtained. In the sequel, the sets of integer, non-negative integer, positive integer, real and complex numbers are denoted by \mathbb{Z} , \mathbb{N} , \mathbb{Z}_+ , \mathbb{R} , \mathbb{C} . We denote $\langle x \rangle = (1 + |x|^2)^{1/2}$ for $x \in \mathbb{R}^d$, $D^\alpha = D_1^{\alpha_1} \dots D_d^{\alpha_d}$, $D_j^{\alpha_j} = i^{-1} \partial^{\alpha_j} / \partial x^{\alpha_j}$, $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_d) \in \mathbb{N}^d$. Finally, fixed $B > 0$ we shall denote by Q_B^c the set of all $(x, \xi) \in \mathbb{R}^{2d}$ for which we have $\langle x \rangle \geq B$ or $\langle \xi \rangle \geq B$.

Following [12], in the sequel we shall consider sequences M_p of positive numbers such that $M_0 = M_1 = 1$ and satisfying all or some of the following conditions:

$$(M.1) \quad M_p^2 \leq M_{p-1} M_{p+1}, \quad p \in \mathbb{Z}_+;$$

$$(M.2) \quad M_p \leq c_0 H^p \min_{0 \leq q \leq p} \{M_{p-q} M_q\}, \quad p, q \in \mathbb{N}, \text{ for some } c_0, H \geq 1;$$

$$(M.3) \quad \sum_{p=q+1}^{\infty} \frac{M_{p-1}}{M_p} \leq c_0 q \frac{M_q}{M_{q+1}}, \quad q \in \mathbb{Z}_+,$$

$$(M.4) \quad \left(\frac{M_p}{p!} \right)^2 \leq \frac{M_{p-1}}{(p-1)!} \cdot \frac{M_{p+1}}{(p+1)!}, \text{ for all } p \in \mathbb{Z}_+,$$

In some assertions in the sequel we could replace (M.3) by the weaker assumption

$$(M.3)' \quad \sum_{p=1}^{\infty} \frac{M_{p-1}}{M_p} < \infty,$$

cf. [12]. It is important to note that (M.4) implies (M.1).

Note that the Gevrey sequence $M_p = p!^s$, $s > 1$, satisfies all of these conditions.

For a multi-index $\alpha \in \mathbb{N}^d$, M_α will mean $M_{|\alpha|}$, $|\alpha| = \alpha_1 + \dots + \alpha_d$. Recall that the associated function for the sequence M_p is defined by

$$M(\rho) = \sup_{p \in \mathbb{N}} \log_+ \frac{\rho^p}{M_p}, \quad \rho > 0.$$

The function $M(\rho)$ is non-negative, continuous, monotonically increasing, it vanishes for sufficiently small $\rho > 0$ and increases more rapidly than $\ln \rho^p$ when ρ tends to infinity, for any $p \in \mathbb{N}$ (see [12]).

For $m > 0$ and a sequence M_p satisfying the conditions (M.1) – (M.3), we shall denote by $\mathcal{S}_\infty^{M_p, m}(\mathbb{R}^d)$ the Banach space of all functions $\varphi \in \mathcal{C}^\infty(\mathbb{R}^d)$ such that

$$\|\varphi\|_m := \sup_{\alpha \in \mathbb{N}^d} \sup_{x \in \mathbb{R}^d} \frac{m^{|\alpha|} |D^\alpha \varphi(x)| e^{M(m|x|)}}{M_\alpha} < \infty, \quad (0.1)$$

endowed with the norm in (0.1) and we denote $\mathcal{S}^{(M_p)}(\mathbb{R}^d) = \varprojlim_{m \rightarrow \infty} \mathcal{S}_\infty^{M_p, m}(\mathbb{R}^d)$ and

$\mathcal{S}^{\{M_p\}}(\mathbb{R}^d) = \varinjlim_{m \rightarrow 0} \mathcal{S}_\infty^{M_p, m}(\mathbb{R}^d)$. In the sequel we shall consider simultaneously the

two latter spaces by using the common notation $\mathcal{S}^*(\mathbb{R}^d)$. For each space we will consider a suitable symbol class. Definitions and statements will be formulated first for the (M_p) case and then for the $\{M_p\}$ case, using the notation $*$. We shall denote by $\mathcal{S}'^*(\mathbb{R}^d)$ the strong dual space of $\mathcal{S}^*(\mathbb{R}^d)$. We refer to [5, 18, 19] for the properties of $\mathcal{S}^*(\mathbb{R}^d)$ and $\mathcal{S}'^*(\mathbb{R}^d)$. Here we just recall that the Fourier transformation is an automorphism on $\mathcal{S}^*(\mathbb{R}^d)$ and on $\mathcal{S}'^*(\mathbb{R}^d)$ and that for $M_p = p!^s$, $s > 1$, we have $M(\rho) \sim \rho^{1/s}$. In this case $\mathcal{S}^*(\mathbb{R}^d)$ coincides respectively with the Gelfand-Shilov

spaces $\Sigma_s(\mathbb{R}^d)$ (resp. $\mathcal{S}_s(\mathbb{R}^d)$) of all functions $\varphi \in \mathcal{C}^\infty(\mathbb{R}^d)$ such that

$$\sup_{\alpha, \beta \in \mathbb{N}^d} h^{-|\alpha| - |\beta|} (\alpha! \beta!)^{-s} \sup_{x \in \mathbb{R}^d} |x^\beta \partial^\alpha \varphi(x)| < \infty$$

for every $h > 0$ (resp. for some $h > 0$), cf. [9, 18].

Following [21] we now introduce the class of pseudodifferential operators to which our results apply. Let M_p, A_p be two sequences of positive numbers. We assume that M_p satisfies (M.1), (M.2) and (M.3) and that A_p satisfies $A_0 = A_1 = 1$, (M.1), (M.2), (M.3)' and (M.4). Moreover we suppose that $A_p \subset M_p$ i.e. there exist $c_0 > 0, L > 0$ such that $A_p \leq c_0 L^p M_p$ for all $p \in \mathbb{N}$. Let $\rho_0 = \inf\{\rho \in \mathbb{R}_+ \mid A_p \subset M_p^\rho\}$. Obviously $0 < \rho_0 \leq 1$. Let $\rho \in \mathbb{R}_+$ be arbitrary but fixed such that $\rho_0 \leq \rho \leq 1$ if the infimum can be reached, or otherwise $\rho_0 < \rho \leq 1$. For any fixed $h > 0, m > 0$ we denote by $\Gamma_{A_p, \rho}^{M_p, \infty}(\mathbb{R}^{2d}; h, m)$ the space of all functions $a(x, \xi) \in \mathcal{C}^\infty(\mathbb{R}^{2d})$ such that

$$\sup_{\alpha, \beta \in \mathbb{Z}_+^d} \sup_{(x, \xi) \in \mathbb{R}^{2d}} \frac{|D_\xi^\alpha D_x^\beta a(x, \xi)| \langle (x, \xi) \rangle^{\rho|\alpha + \beta|} e^{-(M(m|x|) + M(m|\xi|))}}{h^{|\alpha + \beta|} A_\alpha A_\beta} < \infty, \quad (0.2)$$

where $M(\cdot)$ is the associated function for the sequence M_p . Then we define

$$\begin{aligned} \Gamma_{A_p, \rho}^{(M_p), \infty}(\mathbb{R}^{2d}) &= \lim_{m \rightarrow \infty} \lim_{h \rightarrow 0} \Gamma_{A_p, \rho}^{M_p, \infty}(\mathbb{R}^{2d}; h, m); \\ \Gamma_{A_p, \rho}^{\{M_p\}, \infty}(\mathbb{R}^{2d}) &= \lim_{h \rightarrow \infty} \lim_{m \rightarrow 0} \Gamma_{A_p, \rho}^{M_p, \infty}(\mathbb{R}^{2d}; h, m). \end{aligned}$$

Remark 1. We notice that in the case $M_p = p!^s, s > 1$, we can replace $M(m|x|) + M(m|\xi|)$ by $M(m(|x| + |\xi|))$ in (0.2). In particular, in the case of non-quasi-analytic Gelfand-Shilov spaces, we can include symbols of the form $e^{\pm \langle (x, \xi) \rangle^{1/s}}$ in our class, cf. [20].

We associate to any symbol $a \in \Gamma_{A_p, \rho}^{*, \infty}(\mathbb{R}^{2d})$ a pseudodifferential operator $a(x, D)$ defined, as it is usual, by

$$a(x, D)f(x) = (2\pi)^{-d} \int_{\mathbb{R}^d} e^{i\langle x, \xi \rangle} a(x, \xi) \hat{f}(\xi) d\xi, \quad f \in \mathcal{S}^*(\mathbb{R}^d), \quad (0.3)$$

where \hat{f} denotes the Fourier transform of f . In [21] it was proved that operators of the form (0.3) act continuously on $\mathcal{S}^*(\mathbb{R}^d)$ and on $\mathcal{S}^{*'}(\mathbb{R}^d)$. Moreover, a symbolic calculus for $\Gamma_{A_p, \rho}^{*, \infty}(\mathbb{R}^{2d})$ (denoted there by $\Gamma_{A_p, A_p, \rho}^{*, \infty}(\mathbb{R}^{2d})$) has been constructed. As a consequence it was proved that the class of pseudodifferential operators with symbols in $\Gamma_{A_p, \rho}^{*, \infty}(\mathbb{R}^{2d})$ is closed with respect to composition and adjoints. Here we introduce a notion of hypoellipticity for this class.

Definition 0.1. Let $a \in \Gamma_{A_p, \rho}^{*, \infty}(\mathbb{R}^{2d})$. We say that a is $\Gamma_{A_p, \rho}^{*, \infty}$ -hypoelliptic if

- i) there exists $B > 0$ such that there exist $c, m > 0$ (resp. for every $m > 0$ there exists $c > 0$) such that

$$|a(x, \xi)| \geq ce^{-M(m|x|) - M(m|\xi|)}, \quad (x, \xi) \in Q_B^c \quad (0.4)$$

ii) there exists $B > 0$ such that for every $h > 0$ there exists $C > 0$ (resp. there exist $h, C > 0$) such that

$$\left| D_\xi^\alpha D_x^\beta a(x, \xi) \right| \leq C \frac{h^{|\alpha|+|\beta|} |a(x, \xi)| A_\alpha A_\beta}{\langle (x, \xi) \rangle^{\rho(|\alpha|+|\beta|)}}, \quad \alpha, \beta \in \mathbb{N}^d, (x, \xi) \in Q_B^c. \quad (0.5)$$

The main result of the paper is the following

Theorem 0.2. *Let $a \in \Gamma_{A_p, \rho}^{*, \infty}(\mathbb{R}^{2d})$ be $\Gamma_{A_p, \rho}^{*, \infty}$ -hypoelliptic and let $v \in \mathcal{S}^*(\mathbb{R}^d)$. Then every solution $u \in \mathcal{S}'(\mathbb{R}^d)$ to the equation $a(x, D)u = v$ belongs to $\mathcal{S}^*(\mathbb{R}^d)$.*

Remark 2. *Note that the symbols of the form $\langle (x, \xi) \rangle^k$ (for k real) work well as hypoelliptic symbols in the case of the Gevrey sequence $M_p = p!^s, s > 1$. In the case $M_p = p!^s, s > 2$, symbols of the form $e^{\langle (x, \xi) \rangle^{1/s}}$ satisfy the conditions (0.4), (0.5) (cf. [20, Section 5] for other examples of hypoelliptic operators in another context). The more sophisticated analysis for $s > 1$ will be considered separately in a forthcoming paper.*

In [10] the authors characterize Gelfand-Shilov spaces through the Fourier expansions of their elements by the eigenfunctions of a positive globally elliptic Shubin type operator, cf. [24], and the sub-exponential growth with eigenvalues of the corresponding Fourier coefficients. With this, one can verify that the lower bound assumption (0.4) is sharp if we consider operators of the form $\exp(-P^{1/ms})u := \sum_{j=1}^{\infty} e^{-\lambda_j^{1/ms}} u_j \varphi_j$, where P is a positive globally elliptic Shubin differential operator of order m , λ_j are its eigenvalues, $\{\varphi_j\}_{j \in \mathbb{N}}$ is an orthonormal basis of eigenfunctions of P and u_j denote the Fourier coefficients of u .

The proof of Theorem 0.2 is based on the construction of a parametrix for a $\Gamma_{A_p, \rho}^{*, \infty}$ -hypoelliptic operator. To perform this step we use the global calculus developed in [21]. In Section 1 we recall some facts about this calculus. Section 2 is devoted to the construction of the parametrix and to the proof of Theorem 0.2.

1 Pseudodifferential operators on $\mathcal{S}^*(\mathbb{R}^d), \mathcal{S}'(\mathbb{R}^d)$

In this section we recall some facts about the pseudodifferential calculus for operators with symbols in $\Gamma_{A_p, \rho}^{*, \infty}(\mathbb{R}^{2d})$ which will be used in the proofs of the next section. Since the statements below are proved in [21] for slightly more general classes of symbols, we prefer to report here the same results as they should be read for the class $\Gamma_{A_p, \rho}^{*, \infty}(\mathbb{R}^{2d})$ in order to make the paper self-contained. For proofs and further details we refer to [21]. First we recall the notion of asymptotic expansion, cf. [21, Definition 2].

Definition 1.1. *Let M_p and A_p be as in the definition of $\Gamma_{A_p, \rho}^{*, \infty}(\mathbb{R}^{2d})$ and let $m_0 = 0, m_p = M_p/M_{p-1}, p \in \mathbb{Z}_+$. We denote by $FS_{A_p, \rho}^{*, \infty}(\mathbb{R}^{2d})$ the space of all formal sums $\sum_{j \in \mathbb{N}} a_j$ such that for some $B > 0$, $a_j \in \mathcal{C}^\infty(\text{int} Q_{Bm_j}^c)$ and satisfy the following condition: there exists $m > 0$ such that for every $h > 0$ (resp. there exists $h > 0$)*

such that for every $m > 0$) we have

$$\sup_{j \in \mathbb{N}} \sup_{\alpha, \beta \in \mathbb{N}^d} \sup_{(x, \xi) \in Q_{Bm, j}^c} \frac{|D_\xi^\alpha D_x^\beta a_j(x, \xi)| \langle (x, \xi) \rangle^{\rho(|\alpha| + |\beta| + 2j)} e^{-M(m|x|) - M(m|\xi|)}}{h^{|\alpha| + |\beta| + 2j} A_\alpha A_\beta A_j^2} < \infty.$$

Notice that any symbol $a \in \Gamma_{A_p, \rho}^{*, \infty}(\mathbb{R}^{2d})$ can be regarded as an element $\sum_{j \in \mathbb{N}} a_j$ of $FS_{A_p, \rho}^{*, \infty}(\mathbb{R}^{2d})$ with $a_0 = a, a_j = 0$ for $j \geq 1$.

Definition 1.2. A symbol $a \in \Gamma_{A_p, \rho}^{*, \infty}(\mathbb{R}^{2d})$ is equivalent to $\sum_{j \in \mathbb{N}} a_j \in FS_{A_p, \rho}^{*, \infty}(\mathbb{R}^{2d})$ (we write $a \sim \sum_{j \in \mathbb{N}} a_j$ in this case) if there exist $m, B > 0$ such that for every $h > 0$ (resp. there exist $h, B > 0$ such that for every $m > 0$) the following condition holds:

$$\sup_{N \in \mathbb{Z}_+} \sup_{\alpha, \beta \in \mathbb{N}^d} \sup_{(x, \xi) \in Q_{Bm, N}^c} \frac{\left| D_\xi^\alpha D_x^\beta \left(a(x, \xi) - \sum_{j < N} a_j(x, \xi) \right) \right| e^{-M(m|x|) - M(m|\xi|)}}{h^{|\alpha| + |\beta| + 2N} A_\alpha A_\beta A_N^2 \langle (x, \xi) \rangle^{-\rho(|\alpha| + |\beta| + 2N)}} < \infty.$$

In [21] it was proved that if $a \sim 0$, then the operator $a(x, D)$ is $*$ -regularizing, i.e. it extends to a continuous map from $\mathcal{S}'(\mathbb{R}^d)$ to $\mathcal{S}^*(\mathbb{R}^d)$. Moreover we have the following result, cf. [21, Theorem 4].

Proposition 1.3. Let $\sum_{j \in \mathbb{N}} a_j \in FS_{A_p, \rho}^{*, \infty}(\mathbb{R}^{2d})$. Then there exists a symbol $a \in \Gamma_{A_p, \rho}^{*, \infty}(\mathbb{R}^{2d})$ such that $a \sim \sum_{j \in \mathbb{N}} a_j$.

Finally we recall the following composition theorem, cf. [21, Corollary 1].

Theorem 1.4. Let $a, b \in \Gamma_{A_p, \rho}^{*, \infty}(\mathbb{R}^{2d})$ with asymptotic expansions $a \sim \sum_{j \in \mathbb{N}} a_j$ and $b \sim \sum_{j \in \mathbb{N}} b_j$. Then there exists $c \in \Gamma_{A_p, \rho}^{*, \infty}(\mathbb{R}^{2d})$ and a $*$ -regularizing operator T such that $a(x, D)b(x, D) = c(x, D) + T$. Moreover c has the following asymptotic expansion

$$c(x, \xi) \sim \sum_{j \in \mathbb{N}} \sum_{s+k+l=j} \sum_{|\alpha|=l} \frac{1}{\alpha!} \partial_\xi^\alpha a_s(x, \xi) D_x^\alpha b_k(x, \xi).$$

2 Hypoellipticity and parametrix

In this section we construct the symbol of a left (and right) parametrix for a $\Gamma_{A_p, \rho}^{*, \infty}$ -hypoelliptic operator starting from the asymptotic expansion of the symbol and using the symbolic calculus developed in [21]. To do this we need some preliminary results.

Lemma 2.1. Let M_p be a sequence of positive numbers satisfying (M.4) and $M_0 = M_1 = 1$. Then for all $2 \leq q \leq p$, $\left(\frac{M_q}{q!} \right)^{1/(q-1)} \leq \left(\frac{M_p}{p!} \right)^{1/(p-1)}$.

Proof. For brevity in notation put $N_p = M_p/p!$. Then $N_0 = N_1 = 1$ and N_p satisfies (M.1). Moreover the sequence N_{p-1}/N_p is monotonically decreasing. It is enough to prove that $N_p^{1/(p-1)} \leq N_{p+1}^{1/p}$ for $p \geq 2$, $p \in \mathbb{N}$. The proof goes by induction. For $p = 2$ one easily verifies this. Assume that it holds for some $p \geq 2$. Then we have

$$\begin{aligned} N_{p+1}^{2p+2} &\leq N_p^{p+1} N_{p+2}^{p+1} \leq N_p N_{p+1}^{p-1} N_{p+2}^{p+1} = N_{p+2}^{2p} N_p \left(\frac{N_{p+1}}{N_{p+2}} \right)^{p-1} \\ &\leq N_{p+2}^{2p} N_p \frac{N_{p-1}}{N_p} \cdot \dots \cdot \frac{N_1}{N_2} = N_{p+2}^{2p}, \end{aligned}$$

from which the desired inequality follows. \square

Lemma 2.2. *Let M_p satisfy (M.4) and $M_0 = M_1 = 1$. Then for all $\alpha, \beta \in \mathbb{N}^d$ such that $\beta \leq \alpha$ and $1 \leq |\beta| \leq |\alpha| - 1$ the inequality $\binom{\alpha}{\beta} M_{\alpha-\beta} M_\beta \leq |\alpha| M_{|\alpha|-1}$ holds.*

Proof. We will consider two cases.

Case 1. $2 \leq |\beta| \leq |\alpha| - 2$.

If we use Lemma 2.1 and the inequality $\binom{\kappa}{\nu} \leq \binom{|\kappa|}{|\nu|}$ for $\nu \leq \kappa$, $\kappa, \nu \in \mathbb{N}^d$, we have

$$\begin{aligned} \binom{\alpha}{\beta} M_{\alpha-\beta} M_\beta &\leq |\alpha|! \cdot \frac{M_{\alpha-\beta}}{(|\alpha| - |\beta|)!} \cdot \frac{M_\beta}{|\beta|!} \\ &\leq |\alpha|! \cdot \left(\frac{M_{|\alpha|-1}}{(|\alpha| - 1)!} \right)^{\frac{|\alpha| - |\beta| - 1}{|\alpha| - 2}} \cdot \left(\frac{M_{|\alpha|-1}}{(|\alpha| - 1)!} \right)^{\frac{|\beta| - 1}{|\alpha| - 2}} = |\alpha| M_{|\alpha|-1}. \end{aligned}$$

Case 2. $|\beta| = 1$ or $|\beta| = |\alpha| - 1$.

Then obviously $\binom{\alpha}{\beta} M_{\alpha-\beta} M_\beta \leq |\alpha| M_{|\alpha|-1}$. \square

In the following we assume that A_p satisfies the conditions (M.1), (M.2), (M.3)' and (M.4). Furthermore we suppose that $A_0 = A_1 = 1$. Because of (M.3)', $A_p/(pA_{p-1}) \rightarrow \infty$, when $p \rightarrow \infty$, see [12]. Under these assumptions we can prove the following result.

Lemma 2.3. *Let $a \in \Gamma_{A_p, \rho}^{*, \infty}(\mathbb{R}^{2d})$ be $\Gamma_{A_p, \rho}^{*, \infty}$ -hypoelliptic. Then, the function $p_0(x, \xi) = a(x, \xi)^{-1}$ satisfies the following condition: for every $h > 0$ there exists $C > 0$ (resp. there exist $h, C > 0$) such that*

$$\left| D_\xi^\alpha D_x^\beta p_0(x, \xi) \right| \leq C \frac{h^{|\alpha| + |\beta|} |p_0(x, \xi)| A_{\alpha+\beta}}{\langle (x, \xi) \rangle^{\rho(|\alpha| + |\beta|)}}, \quad \alpha, \beta \in \mathbb{N}^d, (x, \xi) \in Q_B^c. \quad (2.1)$$

Proof. We observe preliminary that (M.1) and (M.2) on A_p imply that (0.5) is equivalent to saying that there exists $B > 0$ such that for every $h > 0$ there exists $C > 0$ (resp. there exist $h, C > 0$) such that

$$\left| D_\xi^\alpha D_x^\beta a(x, \xi) \right| \leq C \frac{h^{|\alpha| + |\beta|} |a(x, \xi)| A_{\alpha+\beta}}{\langle (x, \xi) \rangle^{\rho(|\alpha| + |\beta|)}}, \quad \alpha, \beta \in \mathbb{N}^d, (x, \xi) \in Q_B^c. \quad (2.2)$$

Then, to simplify the notation, we set $w = (x, \xi)$. First we will consider the (M_p) case. Let $h > 0$ be arbitrary but fixed and take $h_1 > 0$ such that $2^{4d+2}h_1 \leq h$. Then there exists $C_{h_1} \geq 1$ such that

$$|D_w^\alpha a(w)| \leq C_{h_1} \frac{h_1^{|\alpha|} |a(w)| A_\alpha}{\langle w \rangle^{\rho|\alpha|}}, \quad \alpha \in \mathbb{N}^{2d}, w \in Q_B^c. \quad (2.3)$$

Now, there exists $t \in \mathbb{Z}_+$ such that $C_{h_1} \leq 2^t$. Then, for $|\alpha| \geq t$,

$$|D_w^\alpha a(w)| \leq \frac{(2h_1)^{|\alpha|} |a(w)| A_\alpha}{\langle w \rangle^{\rho|\alpha|}}, \quad w \in Q_B^c. \quad (2.4)$$

Choose $s \in \mathbb{N}$, $s > t + 1$, such that

$$C_{h_1} s' A_{s'-1} \leq A_{s'}, \quad \text{for all } s' \geq s. \quad (2.5)$$

We will prove that

$$|D_w^\alpha p_0(w)| \leq C_{h_1}^{\min\{s, |\alpha|\}} \frac{h^{|\alpha|} |p_0(w)| A_\alpha}{\langle w \rangle^{\rho|\alpha|}}, \quad \alpha \in \mathbb{N}^{2d}, w \in Q_B^c, \quad (2.6)$$

which will complete the proof in the (M_p) case.

For $|\alpha| = 0$, (2.6) is obviously true. Suppose that it is true for $|\alpha| \leq k$, for some $0 \leq k \leq s - 1$. We will prove that it holds for $|\alpha| = k + 1$. If we differentiate the equality $a(w)p_0(w) = 1$ on Q_B^c , we have

$$|a(w)| |D_w^\alpha p_0(w)| \leq \sum_{\substack{\beta \leq \alpha \\ \beta \neq 0}} \binom{\alpha}{\beta} |D_w^{\alpha-\beta} p_0(w)| \cdot |D_w^\beta a(w)|.$$

We can use the inductive hypothesis for the terms $|D_w^{\alpha-\beta} p_0(w)|$, Lemma 2.2 and the fact that $qA_{q-1} \leq A_q$, $\forall q \in \mathbb{Z}_+$, (which follows from (M.4)) to obtain

$$\begin{aligned} |D_w^\alpha p_0(w)| &\leq \frac{C_{h_1}^{k+1} |p_0(w)|}{\langle w \rangle^{\rho|\alpha|}} \sum_{\substack{\beta \leq \alpha \\ \beta \neq 0}} \binom{\alpha}{\beta} h^{|\alpha|-|\beta|} h_1^{|\beta|} A_{\alpha-\beta} A_\beta \\ &\leq \frac{C_{h_1}^{k+1} |p_0(w)| h^{|\alpha|} A_\alpha}{\langle w \rangle^{\rho|\alpha|}} \sum_{\substack{\beta \leq \alpha \\ \beta \neq 0}} \left(\frac{h_1}{h} \right)^{|\beta|} \\ &\leq \frac{C_{h_1}^{k+1} |p_0(w)| h^{|\alpha|} A_\alpha}{\langle w \rangle^{\rho|\alpha|}} \sum_{r=1}^{\infty} \left(\frac{h_1}{h} \right)^r \sum_{|\beta|=r} 1. \end{aligned}$$

Since

$$\sum_{r=1}^{\infty} \left(\frac{h_1}{h} \right)^r \sum_{|\beta|=r} 1 \leq \sum_{r=1}^{\infty} \binom{r+2d-1}{2d-1} \left(\frac{h_1}{h} \right)^r \leq \sum_{r=1}^{\infty} \left(\frac{2^{4d} h_1}{h} \right)^r \leq 1,$$

(2.6) is true for $0 \leq |\alpha| \leq s$. To continue the induction, assume that it is true for $|\alpha| \leq k$, with $k \geq s$. To prove it for $|\alpha| = k + 1$, differentiate the equality $a(w)p_0(w) = 1$ for $w \in Q_B^c$. We obtain

$$|a(w)| |D_w^\alpha p_0(w)| \leq \sum_{\substack{\beta \leq \alpha \\ \beta \neq 0, \beta \neq \alpha}} \binom{\alpha}{\beta} \left| D_w^{\alpha-\beta} p_0(w) \right| \left| D_w^\beta a(w) \right| + |p_0(w)| |D_w^\alpha a(w)|.$$

We can use the inductive hypothesis for the terms $\left| D_w^{\alpha-\beta} p_0(w) \right|$, Lemma 2.2 and (2.5) to obtain

$$\begin{aligned} |D_w^\alpha p_0(w)| &\leq \frac{C_{h_1}^s |p_0(w)|}{\langle w \rangle^{\rho|\alpha|}} \left((2h_1)^{|\alpha|} A_\alpha + \sum_{\substack{\beta \leq \alpha \\ \beta \neq 0, \beta \neq \alpha}} \binom{\alpha}{\beta} C_{h_1} h^{|\alpha|-|\beta|} h_1^{|\beta|} A_{\alpha-\beta} A_\beta \right) \\ &\leq \frac{C_{h_1}^s |p_0(w)|}{\langle w \rangle^{\rho|\alpha|}} \left((2h_1)^{|\alpha|} A_\alpha + \sum_{\substack{\beta \leq \alpha \\ \beta \neq 0, \beta \neq \alpha}} h^{|\alpha|-|\beta|} h_1^{|\beta|} C_{h_1} |\alpha| A_{|\alpha|-1} \right) \\ &\leq \frac{C_{h_1}^s |p_0(w)|}{\langle w \rangle^{\rho|\alpha|}} \left((2h_1)^{|\alpha|} A_\alpha + A_\alpha h^{|\alpha|} \sum_{\substack{\beta \leq \alpha \\ \beta \neq 0, \beta \neq \alpha}} \left(\frac{h_1}{h} \right)^{|\beta|} \right) \\ &\leq \frac{C_{h_1}^s h^{|\alpha|} |p_0(w)| A_\alpha}{\langle w \rangle^{\rho|\alpha|}} \sum_{r=1}^{\infty} \left(\frac{2h_1}{h} \right)^r \sum_{|\beta|=r} 1 \\ &= \frac{C_{h_1}^s h^{|\alpha|} |p_0(w)| A_\alpha}{\langle w \rangle^{\rho|\alpha|}} \sum_{r=1}^{\infty} \binom{r+2d-1}{2d-1} \left(\frac{2h_1}{h} \right)^r. \end{aligned}$$

Finally, we observe that

$$\sum_{r=1}^{\infty} \binom{r+2d-1}{2d-1} \left(\frac{2h_1}{h} \right)^r \leq \sum_{r=1}^{\infty} \left(\frac{2^{4d+1} h_1}{h} \right)^r \leq 1.$$

This completes the induction.

In the $\{M_p\}$ case, there exist $h_1, C_{h_1} > 0$ such that (2.3) holds. Take h such that $2^{4d+2} h_1 \leq h$. Choose t and s as in (2.4) and (2.5). Then we can prove (2.6) in the same way as for the (M_p) case. \square

Remark 3. We observe that to prove Lemma 2.3 we can replace the assumption (M.4) on A_p by a weaker assumption. Namely we can assume that there exists $K > 0$ such that $\left(\frac{M_q}{q!} \right)^{1/q} \leq K \left(\frac{M_p}{p!} \right)^{1/p}$, for all $1 \leq q \leq p$. In fact, the latter condition is the same adopted to prove that $1/f \in \mathcal{E}^*(\mathbb{R})$ when $f \in \mathcal{E}^*(\mathbb{R})$ and $\inf |f(x)| \neq 0$ (cf. [1] for the Beurling case and [23] for the Roumieu case). The proof in [1], [23] relies on careful considerations of the coefficients in the Faà di Bruno formula applied to the composition of the mapping $t \mapsto 1/t$ with $a(x, \xi)$. On the contrary (M.4) is needed to prove the next Lemma 2.4.

Lemma 2.4. *Let $a \in \Gamma_{A_p, \rho}^{*, \infty}(\mathbb{R}^{2d})$ be $\Gamma_{A_p, \rho}^{*, \infty}$ -hypoelliptic. Define $p_0(x, \xi) = a(x, \xi)^{-1}$ and inductively*

$$p_j(x, \xi) = -p_0(x, \xi) \sum_{0 < |\nu| \leq j} \frac{1}{\nu!} \partial_\xi^\nu p_{j-|\nu|}(x, \xi) D_x^\nu a(x, \xi), j \in \mathbb{Z}_+.$$

Then, the functions p_j satisfy the following conditions:

there exist $B > 0$ such that for every $h > 0$ there exists $C > 0$ (resp. there exist $h, C > 0$) such that

$$\left| D_\xi^\alpha D_x^\beta p_j(x, \xi) \right| \leq C \frac{h^{|\alpha|+|\beta|+2j} A_{|\alpha|+|\beta|+2j} |p_0(x, \xi)|}{\langle (x, \xi) \rangle^{\rho(|\alpha|+|\beta|+2j)}}, \quad (2.7)$$

for all $\alpha, \beta \in \mathbb{N}^d$, $(x, \xi) \in Q_B^c$, $j \in \mathbb{Z}_+$;

there exist $m, B > 0$ such that for every $h > 0$ there exists $C > 0$ (resp. there exist $h, B > 0$ such that for every $m > 0$ there exists $C > 0$) such that

$$\left| D_\xi^\alpha D_x^\beta p_j(x, \xi) \right| \leq C \frac{h^{|\alpha|+|\beta|+2j} A_{|\alpha|+|\beta|+2j} e^{M(m|x|)} e^{M(m|\xi|)}}{\langle (x, \xi) \rangle^{\rho(|\alpha|+|\beta|+2j)}}, \quad (2.8)$$

for all $\alpha, \beta \in \mathbb{N}^d$, $(x, \xi) \in Q_B^c$, $j \in \mathbb{Z}_+$.

Proof. First, observe that it is enough to prove (2.7) since (2.8) follows from (2.7) by (0.4) (possibly with different constants). As before, we put $w = (x, \xi)$. We will consider first the (M_p) case. Let $h > 0$ be fixed. Choose $h_1 > 0$ so small such that $2^{9d+1} h_1 \leq h$ and $e^{4^d d h_1 / h} - 1 \leq 1/2$. Then by assumption and Lemma 2.3, there exists $C_{h_1} \geq 1$ such that

$$|D_w^\alpha a(w)| \leq C_{h_1} \frac{h_1^{|\alpha|} |a(w)| A_\alpha}{\langle w \rangle^{\rho|\alpha|}}, \quad \alpha \in \mathbb{N}^{2d}, w \in Q_B^c, \quad (2.9)$$

$$|D_w^\alpha p_0(w)| \leq C_{h_1} \frac{h_1^{|\alpha|} |p_0(w)| A_\alpha}{\langle w \rangle^{\rho|\alpha|}}, \quad \alpha \in \mathbb{N}^{2d}, w \in Q_B^c, \quad (2.10)$$

Take $s \in \mathbb{Z}_+$, such that

$$C_{h_1}^2 s' A_{s'-1} \leq A_{s'}, \quad \text{for all } s' \geq s. \quad (2.11)$$

We will prove that, for $j \geq 1$,

$$|D_w^\alpha p_j(w)| \leq C_{h_1}^{2 \min\{s, j\} + 1} \frac{h^{|\alpha|+2j} A_{|\alpha|+2j} |p_0(w)|}{\langle w \rangle^{\rho(|\alpha|+2j)}}, \quad (2.12)$$

for all $\alpha \in \mathbb{N}^{2d}$, $w \in Q_B^c$, $j \in \mathbb{Z}_+$, which will prove the lemma in the (M_p) case. We can argue by induction on j . For $j = 1$, we have

$$|D_w^\alpha p_1(w)| \leq \sum_{\beta+\gamma+\delta=\alpha} \sum_{|\nu|=1} \frac{\alpha!}{\beta! \gamma! \delta!} \left| D_w^\beta p_0(w) \right| \left| D_w^\gamma D_\xi^\nu p_0(w) \right| \left| D_w^\delta D_x^\nu a(w) \right|$$

$$\leq \frac{C_{h_1}^3 |p_0(w)|}{\langle w \rangle^{\rho(|\alpha|+2)}} \sum_{\beta+\gamma+\delta=\alpha} \frac{d \cdot \alpha!}{\beta! \gamma! \delta!} h_1^{|\beta|} A_{|\beta|} h^{|\gamma|+1} A_{|\gamma|+1} h_1^{|\delta|+1} A_{|\delta|+1}.$$

For $|\gamma| \geq 1$, by using Lemma 2.1, we obtain

$$A_{|\gamma|+1} \leq (|\gamma| + 1)! \left(\frac{A_{|\alpha|+2}}{(|\alpha| + 2)!} \right)^{\frac{|\gamma|}{|\alpha|+1}}.$$

For $|\gamma| = 0$ this trivially holds. Also, if $|\beta| \geq 2$,

$$A_\beta \leq |\beta|! \left(\frac{A_{|\alpha|+2}}{(|\alpha| + 2)!} \right)^{\frac{|\beta|-1}{|\alpha|+1}} \leq |\beta|! \left(\frac{A_{|\alpha|+2}}{(|\alpha| + 2)!} \right)^{\frac{|\beta|}{|\alpha|+1}}$$

and this obviously holds if $|\beta| = 1$ or $|\beta| = 0$ (note that (M.4) implies that $A_p \geq p!$ for all $p \in \mathbb{N}$). Moreover for $|\delta| \geq 1$, by Lemma 2.1, we have

$$A_{|\delta|+1} \leq (|\delta| + 1)! \left(\frac{A_{|\alpha|+2}}{(|\alpha| + 2)!} \right)^{\frac{|\delta|}{|\alpha|+1}}.$$

If $|\delta| = 0$ this inequality obviously holds. Insert these inequalities in the estimate for $|D_w^\alpha p_1(w)|$ to obtain

$$\begin{aligned} |D_w^\alpha p_1(w)| &\leq \frac{C_{h_1}^3 h^{|\alpha|+2} A_{|\alpha|+2} |p_0(w)|}{\langle w \rangle^{\rho(|\alpha|+2)}} \sum_{\beta+\gamma+\delta=\alpha} \frac{d \cdot \alpha!}{\beta! \gamma! \delta!} \left(\frac{h_1}{h} \right)^{|\beta|+|\delta|+1} \\ &\quad \cdot \frac{(|\gamma| + 1)! |\beta|! (|\delta| + 1)!}{(|\alpha| + 2)!}. \end{aligned}$$

Observe that

$$\begin{aligned} \frac{\alpha!}{\beta! \gamma! \delta!} &= \binom{\alpha}{\beta+\gamma} \binom{\beta+\gamma}{\beta} \leq \binom{|\alpha|}{|\beta+\gamma|} \binom{|\beta+\gamma|}{|\beta|} \\ &= \frac{|\alpha|!}{|\beta|! |\gamma|! |\delta|!} \leq \frac{(|\alpha| + 1)!}{|\beta|! (|\gamma| + 1)! |\delta|!} \leq \frac{(|\alpha| + 2)!}{|\beta|! (|\gamma| + 1)! (|\delta| + 1)!}. \end{aligned}$$

We obtain

$$|D_w^\alpha p_1(w)| \leq \frac{C_{h_1}^3 h^{|\alpha|+2} A_{|\alpha|+2} |p_0(w)|}{\langle w \rangle^{\rho(|\alpha|+2)}} \sum_{\beta+\gamma+\delta=\alpha} \left(\frac{2^d h_1}{h} \right)^{|\beta|+|\delta|+1}.$$

Note that

$$\begin{aligned} \sum_{\beta+\gamma+\delta=\alpha} \left(\frac{2^d h_1}{h} \right)^{|\beta|+|\delta|+1} &\leq \sum_{l=0}^{\infty} \sum_{|\beta|+|\delta|=l} \left(\frac{2^d h_1}{h} \right)^{l+1} \\ &\leq \sum_{l=0}^{\infty} \binom{l+4d-1}{4d-1} \left(\frac{2^d h_1}{h} \right)^{l+1} \end{aligned}$$

$$\leq \sum_{l=0}^{\infty} \left(\frac{2^{9d} h_1}{h} \right)^{l+1} \leq 1,$$

which completes the proof for $j = 1$. Suppose that it holds for all $j \leq k$, $k \leq s - 1$, $k \in \mathbb{Z}_+$. We will prove it for $j = k + 1$.

$$\begin{aligned} |D_w^\alpha p_j(w)| &\leq \sum_{\beta+\gamma+\delta=\alpha} \sum_{0 < |\nu| \leq j} \frac{\alpha!}{\beta! \gamma! \delta!} \cdot \frac{1}{\nu!} |D_w^\beta p_0(w)| \cdot |D_w^\gamma D_\xi^\nu p_{j-|\nu|}(w)| \cdot |D_w^\delta D_x^\nu a(w)| \\ &\leq \frac{C_{h_1}^{2j+1} |p_0(w)|}{\langle w \rangle^{\rho(|\alpha|+2j)}} \sum_{\beta+\gamma+\delta=\alpha} \sum_{0 < |\nu| \leq j} \frac{\alpha!}{\beta! \gamma! \delta! \nu!} \cdot h_1^{|\beta|} A_{|\beta|} h^{|\gamma|+2j-|\nu|} A_{|\gamma|+2j-|\nu|} h_1^{|\delta|+|\nu|} A_{|\delta|+|\nu|}, \end{aligned}$$

where we used the inductive hypothesis for the derivatives of the terms $p_{j-|\nu|}(w)$. By using Lemma 2.1, we obtain (note that $2j - |\nu| \geq 2$)

$$\begin{aligned} A_{|\gamma|+2j-|\nu|} &\leq (|\gamma| + 2j - |\nu|)! \left(\frac{A_{|\alpha|+2j}}{(|\alpha| + 2j)!} \right)^{\frac{|\gamma|+2j-|\nu|-1}{|\alpha|+2j-1}} \\ &\leq (|\gamma| + 2j - |\nu|)! \left(\frac{A_{|\alpha|+2j}}{(|\alpha| + 2j)!} \right)^{\frac{|\gamma|+2j-|\nu|}{|\alpha|+2j-1}}, \end{aligned}$$

where the last inequality follows from $A_p \geq p!$, $p \in \mathbb{N}$, which in turn follows from (M.4). Also, if $|\beta| \geq 2$,

$$A_\beta \leq |\beta|! \left(\frac{A_{|\alpha|+2j}}{(|\alpha| + 2j)!} \right)^{\frac{|\beta|-1}{|\alpha|+2j-1}} \leq |\beta|! \left(\frac{A_{|\alpha|+2j}}{(|\alpha| + 2j)!} \right)^{\frac{|\beta|}{|\alpha|+2j-1}}$$

and this obviously holds if $|\beta| = 1$ or $|\beta| = 0$. Moreover for $|\delta| \geq 1$, by Lemma 2.1 (because $|\nu| \geq 1$), we have

$$A_{|\delta|+|\nu|} \leq (|\delta| + |\nu|)! \left(\frac{A_{|\alpha|+2j}}{(|\alpha| + 2j)!} \right)^{\frac{|\delta|+|\nu|-1}{|\alpha|+2j-1}}.$$

If $|\delta| = 0$ and $|\nu| \geq 2$ Lemma 2.1 implies the same inequality and if $|\delta| = 0$ and $|\nu| = 1$ this inequality obviously holds. If we insert these inequalities in the estimate for $|D_w^\alpha p_j(w)|$, we obtain

$$\begin{aligned} |D_w^\alpha p_j(w)| &\leq \frac{C_{h_1}^{2j+1} |p_0(w)|}{\langle w \rangle^{\rho(|\alpha|+2j)}} \sum_{\beta+\gamma+\delta=\alpha} \sum_{0 < |\nu| \leq j} \frac{\alpha!}{\beta! \gamma! \delta! \nu!} h_1^{|\beta|} h^{|\gamma|+2j-|\nu|} h_1^{|\delta|+|\nu|} \\ &\quad \cdot (|\gamma| + 2j - |\nu|)! \left(\frac{A_{|\alpha|+2j}}{(|\alpha| + 2j)!} \right)^{\frac{|\gamma|+2j-|\nu|}{|\alpha|+2j-1}} |\beta|! \left(\frac{A_{|\alpha|+2j}}{(|\alpha| + 2j)!} \right)^{\frac{|\beta|}{|\alpha|+2j-1}} \\ &\quad (\delta + |\nu|)! \left(\frac{A_{|\alpha|+2j}}{(|\alpha| + 2j)!} \right)^{\frac{|\delta|+|\nu|-1}{|\alpha|+2j-1}} \end{aligned}$$

$$\begin{aligned}
&= \frac{C_{h_1}^{2j+1} h^{|\alpha|+2j} A_{|\alpha|+2j} |p_0(w)|}{\langle w \rangle^{\rho(|\alpha|+2j)}} \sum_{\beta+\gamma+\delta=\alpha} \sum_{0 < |\nu| \leq j} \frac{\alpha!}{\beta! \gamma! \delta! \nu!} \left(\frac{h_1}{h} \right)^{|\beta|+|\delta|+|\nu|} \\
&\quad \cdot \frac{(|\gamma|+2j-|\nu|)! |\beta|! (|\delta|+|\nu|)!}{(|\alpha|+2j)!}.
\end{aligned}$$

Similarly as above, we have

$$\begin{aligned}
\frac{\alpha!}{\beta! \gamma! \delta!} &\leq \frac{|\alpha|!}{|\beta|! |\gamma|! |\delta|!} \leq \frac{(|\alpha|+2j-|\nu|)!}{|\beta|! (|\gamma|+2j-|\nu|)! |\delta|!} \\
&\leq \frac{(|\alpha|+2j)!}{|\beta|! (|\gamma|+2j-|\nu|)! (|\delta|+|\nu|)!}.
\end{aligned}$$

We obtain

$$|D_w^\alpha p_j(w)| \leq \frac{C_{h_1}^{2j+1} h^{|\alpha|+2j} A_{|\alpha|+2j} |p_0(w)|}{\langle w \rangle^{\rho(|\alpha|+2j)}} \sum_{\beta+\gamma+\delta=\alpha} \sum_{r=1}^{\infty} \sum_{|\nu|=r} \frac{1}{\nu!} \left(\frac{h_1}{h} \right)^{|\beta|+|\delta|+r}.$$

We have the estimate

$$\begin{aligned}
&\sum_{\beta+\gamma+\delta=\alpha} \sum_{r=1}^{\infty} \sum_{|\nu|=r} \frac{1}{\nu!} \left(\frac{h_1}{h} \right)^{|\beta|+|\delta|+r} \\
&\leq \sum_{\beta+\gamma+\delta=\alpha} \sum_{r=1}^{\infty} \binom{r+d-1}{d-1} \frac{d^r}{r!} \left(\frac{h_1}{h} \right)^{|\beta|+|\delta|+r} \\
&\leq \sum_{\beta+\gamma+\delta=\alpha} \left(\frac{h_1}{h} \right)^{|\beta|+|\delta|} \sum_{r=1}^{\infty} \frac{1}{r!} \left(\frac{2^{2d} d h_1}{h} \right)^r \\
&= \left(e^{4^d d h_1/h} - 1 \right) \sum_{\beta+\gamma+\delta=\alpha} \left(\frac{h_1}{h} \right)^{|\beta|+|\delta|} = \left(e^{4^d d h_1/h} - 1 \right) \sum_{\beta+\delta \leq \alpha} \left(\frac{h_1}{h} \right)^{|\beta|+|\delta|} \\
&\leq \left(e^{4^d d h_1/h} - 1 \right) \sum_{l=0}^{\infty} \left(\frac{h_1}{h} \right)^l \sum_{|\beta|+|\delta|=l} 1 \\
&= \left(e^{4^d d h_1/h} - 1 \right) \sum_{l=0}^{\infty} \left(\frac{h_1}{h} \right)^l \binom{l+4d-1}{4d-1} \\
&\leq \left(e^{4^d d h_1/h} - 1 \right) \sum_{l=0}^{\infty} \left(\frac{2^{8d} h_1}{h} \right)^l \leq 1.
\end{aligned}$$

Hence, we proved (2.12) for $1 \leq j \leq s$. Suppose that it holds for all $j \leq k$, $k \geq s$. For $j = k+1$, similarly as above, we obtain

$$\begin{aligned}
|D_w^\alpha p_j(w)| &\leq \frac{C_{h_1}^{2s+1} |p_0(w)|}{\langle w \rangle^{\rho(|\alpha|+2j)}} \sum_{\beta+\gamma+\delta=\alpha} \sum_{0 < |\nu| \leq j} \frac{\alpha!}{\beta! \gamma! \delta! \nu!} \\
&\quad \cdot C_{h_1}^2 h_1^{|\beta|} A_{|\beta|} h^{|\gamma|+2j-|\nu|} A_{|\gamma|+2j-|\nu|} h_1^{|\delta|+|\nu|} A_{|\delta|+|\nu|}.
\end{aligned}$$

Note that $|\gamma| + 2j - |\nu| \geq s$, so, by (2.11), we have

$$C_{h_1}^2 A_{|\gamma|+2j-|\nu|} \leq A_{|\gamma|+2j-|\nu|+1} / (|\gamma| + 2j - |\nu| + 1).$$

Also $|\gamma| + 2j - |\nu| + 1 \leq |\alpha| + 2j$, hence Lemma 2.1 implies

$$C_{h_1}^2 A_{|\gamma|+2j-|\nu|} \leq \frac{A_{|\gamma|+2j-|\nu|+1}}{|\gamma| + 2j - |\nu| + 1} \leq (|\gamma| + 2j - |\nu|)! \left(\frac{A_{|\alpha|+2j}}{(|\alpha| + 2j)!} \right)^{\frac{|\gamma|+2j-|\nu|}{|\alpha|+2j-1}}.$$

In the same manner as above we obtain

$$A_\beta \leq |\beta|! \left(\frac{A_{|\alpha|+2j}}{(|\alpha| + 2j)!} \right)^{\frac{|\beta|}{|\alpha|+2j-1}} \quad \text{and} \quad A_{|\delta|+|\nu|} \leq (|\delta| + |\nu|)! \left(\frac{A_{|\alpha|+2j}}{(|\alpha| + 2j)!} \right)^{\frac{|\delta|+|\nu|-1}{|\alpha|+2j-1}}.$$

If we insert these inequalities in the estimate for $|D_w^\alpha p_j(w)|$ and use the above inequality for $\frac{\alpha!}{\beta! \gamma! \delta!}$ we obtain

$$|D_w^\alpha p_j(w)| \leq \frac{C_{h_1}^{2s+1} h^{|\alpha|+2j} A_{|\alpha|+2j} |p_0(w)|}{\langle w \rangle^{\rho(|\alpha|+2j)}} \sum_{\beta+\gamma+\delta=\alpha} \sum_{r=1}^{\infty} \sum_{|\nu|=r} \frac{1}{\nu!} \left(\frac{h_1}{h} \right)^{|\beta|+|\delta|+r}.$$

We already proved that $\sum_{\beta+\gamma+\delta=\alpha} \sum_{r=1}^{\infty} \sum_{|\nu|=r} \frac{1}{\nu!} \left(\frac{h_1}{h} \right)^{|\beta|+|\delta|+r} \leq 1$, hence the proof for the (M_p) case is complete.

Next, we consider the $\{M_p\}$ case. By assumption and Lemma 2.3, there exist $h_1, C_{h_1} \geq 1$ such that (2.9) and (2.10) hold. Take h so large such that $2^{9d+1} h_1 \leq h$ and $e^{4^d d h_1/h} - 1 \leq 1/2$. There exists $s \in \mathbb{Z}_+$ such that $C_{h_1}^2 s' A_{s'-1} \leq A_{s'}$, for all $s' \geq s$. One proves that

$$|D_w^\alpha p_j(w)| \leq C_{h_1}^{2\min\{s,j\}+1} \frac{h^{|\alpha|+2j} A_{|\alpha|+2j} |p_0(w)|}{\langle w \rangle^{\rho(|\alpha|+2j)}},$$

for all $\alpha \in \mathbb{N}^{2d}$, $w \in Q_B^c$, $j \in \mathbb{Z}_+$, by induction on j in the same manner as for (2.12) in the (M_p) case. This completes the proof in the $\{M_p\}$ case. \square

Theorem 2.5. *Let $a \in \Gamma_{A_p, \rho}^{*, \infty}(\mathbb{R}^{2d})$ be $\Gamma_{A_p, \rho}^{*, \infty}$ -hypoelliptic. Then there exist *-regularizing operators T and T' and $b, b' \in \Gamma_{A_p, \rho}^{*, \infty}(\mathbb{R}^{2d})$ such that $b(x, D)a(x, D) = \text{Id} + T$ and $a(x, D)b'(x, D) = \text{Id} + T'$.*

Proof. Let p_j , $j \in \mathbb{N}$, be as in Lemma 2.4. Then the functions p_0 and p_j , $j \in \mathbb{Z}_+$, satisfy the estimates given in Lemmas 2.3 and 2.4. Since A_p satisfies (M.1) and (M.2), these estimates are equivalent to the following:

there exist $m, B > 0$ such that for every $h > 0$ there exists $C > 0$ (resp. there exist $h, B > 0$ such that for every $m > 0$ there exists $C > 0$) such that

$$\left| D_\xi^\alpha D_x^\beta p_j(x, \xi) \right| \leq C \frac{h^{|\alpha|+|\beta|+2j} A_\alpha A_\beta A_j^2 e^{M(m|x|)} e^{M(m|\xi|)}}{\langle (x, \xi) \rangle^{\rho(|\alpha|+|\beta|+2j)}}, \quad (2.13)$$

for all $\alpha, \beta \in \mathbb{N}^d$, $(x, \xi) \in Q_B^c$, $j \in \mathbb{N}$. One can modify p_0 near the boundary of Q_B^c so that it can be extended to C^∞ function on \mathbb{R}^{2d} and satisfy (2.13) on the whole \mathbb{R}^{2d} . Hence, (2.13) remains true for all $j \in \mathbb{Z}_+$ with larger B . We obtain $\sum_{j=0}^\infty p_j \in FS_{A_p, \rho}^{\infty, *}(\mathbb{R}^{2d})$. Let $b \sim \sum_j p_j$, $b \in \Gamma_{A_p, \rho}^{*, \infty}(\mathbb{R}^{2d})$. By Theorem 1.4 there exist $c \in \Gamma_{A_p, \rho}^{*, \infty}(\mathbb{R}^{2d})$ and a *-regularizing operator \tilde{T}'_1 such that $b(x, D)a(x, D) = c(x, D) + \tilde{T}$ and c has the asymptotic expansion $c \sim \sum_j c_j$, where

$$c_j(x, \xi) = \sum_{s+l=j} \sum_{|\nu|=l} \frac{1}{\nu!} \partial_\xi^\nu p_s(x, \xi) D_x^\nu a(x, \xi).$$

One easily verifies that $c_0(x, \xi) = 1$ on Q_B^c . Also, for $j \in \mathbb{Z}_+$,

$$c_j = p_j a + \sum_{l=1}^j \sum_{|\nu|=l} \frac{1}{\nu!} \partial_\xi^\nu p_{j-l} \cdot D_x^\nu a = p_j a + \sum_{0 < |\nu| \leq j} \frac{1}{\nu!} \partial_\xi^\nu p_{j-|\nu|} \cdot D_x^\nu a = 0,$$

on Q_B^c , by the definition of p_j . Hence, $b(x, D)a(x, D) = \text{Id} + T$ for some *-regularizing operator T . With similar constructions one obtains b' such that $a(x, D)b'(x, D) = \text{Id} + T'$, where T' is a *-regularizing operator. \square

Proof of Theorem 0.2. Let $u \in \mathcal{S}'(\mathbb{R}^d)$ be a solution of $a(x, D)u = v \in \mathcal{S}^*(\mathbb{R}^d)$. Then, applying the left parametrix $b(x, D)$ of $a(x, D)$, we obtain $u = b(x, D)v - Tu$ for some *-regularizing operator T . Hence $u \in \mathcal{S}^*(\mathbb{R}^d)$. The theorem is proved. \square

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